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Blow-up conditions for nonlinear Volterra integral equations with power nonlinearity

Tomasz Małolepszy[a], Wojciech Okrasiński[b,*]

[a] *Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Podgórna 50, 65-246 Zielona Góra, Poland*
[b] *Institute of Mathematics and Computer Science, Wroclaw University of Technology, Wybrzeze Wyspianskiego 27, 50-370 Wroclaw, Poland*

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Abstract

In this work we consider nonlinear Volterra equations of a special type. We find necessary and sufficient conditions for blow-up of solutions to this class of equations.

Keywords: Volterra; Nonlinear integral equations; Power nonlinearity; Blow-up

1. Introduction

We consider the following equation:

$$u(t) = \int_0^t k(t-s)g(u(s))\mathrm{d}s, \quad t \geq 0, \tag{1}$$

where $k : (0,\infty) \to [0,\infty)$ is a locally integrable positive function and $g : [0,\infty) \to [0,\infty)$ is a strictly increasing absolutely continuous function that satisfies the following conditions:

$$g(0) = 0 \quad \text{and} \quad u/g(u) \to 0 \quad \text{as } u \to 0^+. \tag{2}$$

Let $K(t) = \int_0^t k(s)\mathrm{d}s$ and $\lim_{t\to\infty} K(t) = \infty$.

We assume that function $g(t)$ in Eq. (1) has the following form:

$$g(t) = \begin{cases} g_1(t), & t \in [0,d) \\ bt^p, & t \geq d, \end{cases} \tag{3}$$

where $d > 0$, $b > 0$, $p > 1$ and function $g_1(t)$ is taken such that the function $g(t)$ given by (3) satisfies conditions (2). The class of integral equations (1) with nonlinearity of the form (3) is motivated by certain models of explosion phenomena in a diffusive medium; e.g. [1–3].

* Corresponding author. Tel.: +48 71 320 31 48; fax: +48 71 328 07 51.
E-mail address: Wojciech.Okrasinski@pwr.wroc.pl (W. Okrasiński).

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Clearly, $u \equiv 0$ is the trivial solution to (1). In order to examine blow-up solutions, we must consider the case when nontrivial solutions exist. The existence of such solutions follows from additional conditions [4–10].

The aim of our work is to find a necessary and sufficient condition for the existence of blow-up solutions to Eq. (1) with the function $g(t)$ given by (3).

2. The main result

Theorem 2.1 (*Integral Necessary and Sufficient Condition for the Existence of Blow-up*). *Eq. (1) with function $g(t)$ given by (3) has a blow-up solution if and only if the integral*

$$\int_0^t K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s)} \tag{4}$$

is convergent for any $t \in (0, 1)$.

3. Auxiliary theorems and lemmas

To proof Theorem 2.1 we use the following results from [11].

Theorem 3.1 (*Sufficient Condition for the Existence of Blow-up*). *If $w(t)$ is a strictly increasing, positive and continuous function such that $w(t) < g(t)$ for $t \in (a, \infty)$, $w(a) = g(a)$, $t/w(t) \to 0$ as $t \to \infty$ and the series*

$$\sum_{i=0}^{\infty} K^{-1}\left(\frac{(w^{-1}\circ g)^i(t)}{w\left(\left(w^{-1}\circ g\right)^i(t)\right)}\right) \tag{5}$$

is convergent for some point $t \in (a, \infty)$, then a blow-up solution to (1) exists.

Lemma 3.2. *If Eq. (1) has blow-up, $w(t)$ is a strictly increasing, positive and continuous function such that $w(t) < g(t)$ for $t \in (a, \infty)$ and $w(a) = g(a)$, and that $w(t)/t \to 0$ as $t \to \infty$, then for any $\varepsilon \in (0, 1)$ there exists $c_\varepsilon \geq 0$ such that for any $t \in (c_\varepsilon, \infty)$,*

$$K\left(u^{-1}(t) - u^{-1}\left(g^{-1}(w(t))\right)\right) \geq \frac{t^{1-\varepsilon}}{g(t)}. \tag{6}$$

Theorem 3.3 (*Necessary Condition for the Existence of Blow-up*). *If Eq. (1) has a blow-up solution, $w(t)$ is a strictly increasing, positive and continuous function such that $w(t) < g(t)$ for $t \in (a, \infty)$ and $w(a) = g(a)$, and that $w(t)/t \to 0$ as $t \to \infty$, then for any $\varepsilon \in (0, 1)$ and for any $t \in (c_\varepsilon, \infty)$, the series*

$$\sum_{i=0}^{\infty} K^{-1}\left(\frac{\left((w^{-1}\circ g)^i(t)\right)^{1-\varepsilon}}{g\left(\left(w^{-1}\circ g\right)^i(t)\right)}\right) \tag{7}$$

is convergent.

Now we define a family of functions $\phi_a(t)$ and $\psi_a(t)$ through the following formulas:

$$\phi_a(t) = \begin{cases} \phi_1(t), & t \in [0, a) \\ ba^{p-q}t^q, & t \geq a, \end{cases} \tag{8}$$

$$\psi_a(t) = \begin{cases} \psi_1(t), & t \in [0, a) \\ ba^{p-r}t^r, & t \geq a, \end{cases} \tag{9}$$

where a is an arbitrary number from $[d, \infty)$, $p > q > 1 > r > 0$ and functions ϕ_1 and ψ_1 are taken such that the functions ϕ_a and ψ_a satisfy adequately the assumptions of Theorems 3.1 and 3.3 concerning the function $w(t)$. In this case Theorem 3.1 has the following form.

Theorem 3.4 (*Sufficient Condition for the Existence of Blow-up*). *Let the function $g(t)$ in Eq. (1) be given by (3) and the function $w(t)$ be given by (8) for some $a \in [d, \infty)$. If the series*

$$\sum_{i=0}^{\infty} K^{-1}\left(\frac{a^{1-p}}{b}\left(\frac{t}{a}\right)^{(1-q)(\frac{p}{q})^i}\right) \tag{10}$$

is convergent for some $t \in (a, \infty)$, then a blow-up solution to (1) exists.

See also Theorem 3.3.

Theorem 3.5 (*Necessary condition for the existence of blow-up*). *Let the function $g(t)$ in Eq. (1) be given by (3) and the function $w(t)$ be given by (8) for some $a \in [d, \infty)$. If Eq. (1) has a blow-up solution, then for any $\varepsilon \in (0, 1)$ and for any $t \in (c_\varepsilon, \infty)$ (where $c_\varepsilon > a$), the series*

$$\sum_{i=0}^{\infty} K^{-1}\left(\frac{a^{1-\varepsilon-p}}{b}\left(\frac{t}{a}\right)^{(1-\varepsilon-p)(\frac{p}{r})^i}\right)$$

is convergent.

4. Proof of Theorem 2.1

Proof. The sufficient part of the theorem.

Let $\Phi(t) = t^{\frac{p}{q}}$, where $t > a$. We define a sequence $t_0 = \frac{t}{a}$, $t_{n+1} = \Phi(t_n)$. We note that $t_n = \left(\frac{t}{a}\right)^{(\frac{p}{q})^n}$, so this sequence is increasing and divergent to ∞. Since the function $K^{-1}(\frac{a^{1-p}}{b}s^{1-q})$ is decreasing for $s > 1$, this implies that

$$\int_{t_n}^{t_{n+1}} K^{-1}\left(\frac{a^{1-p}}{b}s^{1-q}\right)\frac{\mathrm{d}s}{s\ln s} \geq K^{-1}\left(\frac{a^{1-p}}{b}t_{n+1}^{1-q}\right)\ln\left(\frac{\ln t_{n+1}}{\ln t_n}\right)$$

and finally

$$\int_{t_n}^{t_{n+1}} K^{-1}\left(\frac{a^{1-p}}{b}s^{1-q}\right)\frac{\mathrm{d}s}{s\ln s} \geq \ln\frac{p}{q}K^{-1}\left(\frac{a^{1-p}}{b}t_{n+1}^{1-q}\right).$$

Hence

$$\int_{t_0}^{t_{n+1}} K^{-1}\left(\frac{a^{1-p}}{b}s^{1-q}\right)\frac{\mathrm{d}s}{s\ln s} \geq \ln\frac{p}{q}\sum_{i=1}^{n+1} K^{-1}\left(\frac{a^{1-p}}{b}\left(\frac{t}{a}\right)^{(1-q)(\frac{p}{q})^i}\right).$$

Let $n \to \infty$. Then we obtain

$$\int_{\frac{t}{a}}^{\infty} K^{-1}\left(\frac{a^{1-p}}{b}s^{1-q}\right)\frac{\mathrm{d}s}{s\ln s} \geq \ln\frac{p}{q}\sum_{i=1}^{\infty} K^{-1}\left(\frac{a^{1-p}}{b}\left(\frac{t}{a}\right)^{(1-q)(\frac{p}{q})^i}\right).$$

Because a is arbitrary, we can choose a such that $\frac{a^{1-p}}{b} < 1$. This gives

$$\int_{\frac{t}{a}}^{\infty} K^{-1}\left(s^{1-q}\right)\frac{\mathrm{d}s}{s\ln s} > \int_{\frac{t}{a}}^{\infty} K^{-1}\left(\frac{a^{1-p}}{b}s^{1-q}\right)\frac{\mathrm{d}s}{s\ln s}.$$

We observe that

$$\int_{\frac{t}{a}}^{\infty} K^{-1}\left(s^{1-q}\right)\frac{\mathrm{d}s}{s\ln s} = \int_{0}^{(\frac{t}{a})^{1-q}} K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s)}.$$

This means that the convergence of the integral $\int_{\frac{t}{a}}^{\infty} K^{-1}\left(s^{1-q}\right)\frac{\mathrm{d}s}{s\ln s}$, where $t \in (a, \infty)$, is equivalent to the convergence of the integral $\int_0^t K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s)}$, where $t \in (0, 1)$. From the assumptions and Theorem 3.4 we get our assertion.

The necessity part of the theorem.

Let $\Psi(t) = t^{\frac{p}{r}}$, where $t > c_\varepsilon$. We define a sequence $t_0 = \frac{t}{a}$, $t_{n+1} = \Psi(t_n)$. Because $t_n = (\frac{t}{a})^{(\frac{p}{r})^n}$, we see that the sequence is increasing and divergent to ∞. We have

$$\int_{t_n}^{t_{n+1}} K^{-1}\left(\frac{a^{1-\varepsilon-p}}{b} s^{1-\varepsilon-p}\right)\frac{\mathrm{d}s}{s\ln s} \le \ln\frac{p}{r} K^{-1}\left(\frac{a^{1-\varepsilon-p}}{b} t_n^{1-\varepsilon-p}\right).$$

Therefore

$$\int_{t_0}^{t_{n+1}} K^{-1}\left(\frac{a^{1-\varepsilon-p}}{b} s^{1-\varepsilon-p}\right)\frac{\mathrm{d}s}{s\ln s} \le \ln\frac{p}{r}\sum_{i=0}^{n} K^{-1}\left(\frac{a^{1-\varepsilon-p}}{b}\left(\frac{t}{a}\right)^{(1-\varepsilon-p)(\frac{p}{r})^i}\right).$$

From the last inequality, as $n \to \infty$, we get

$$\int_{\frac{t}{a}}^{\infty} K^{-1}\left(\frac{a^{1-\varepsilon-p}}{b} s^{1-\varepsilon-p}\right)\frac{\mathrm{d}s}{s\ln s} \le \ln\frac{p}{r}\sum_{i=0}^{\infty} K^{-1}\left(\frac{a^{1-\varepsilon-p}}{b}\left(\frac{t}{a}\right)^{(1-\varepsilon-p)(\frac{p}{r})^i}\right).$$

The following equality is true:

$$\int_{\frac{t}{a}}^{\infty} K^{-1}\left(\frac{a^{1-\varepsilon-p}}{b} s^{1-\varepsilon-p}\right)\frac{\mathrm{d}s}{s\ln s} = \int_0^{\frac{a^{1-\varepsilon-p}}{b}(\frac{t}{a})^{1-\varepsilon-p}} K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s + \ln\frac{a^{1-\varepsilon-p}}{b})}.$$

Because

$$\lim_{t\to 0^+}\frac{-\ln t + \ln\frac{a^{1-\varepsilon-p}}{b}}{-\ln t} = 1,$$

we have that $-\ln t + \ln\frac{a^{1-\varepsilon-p}}{b} \sim -\ln t$ as $t \to 0^+$. This fact, the last equality and Theorem 3.5 mean that the integral

$$\int_0^t K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s)}$$

is convergent for any $t \in (0, \delta)$ for some $\delta > 0$, which completes the proof. ■

5. Conclusions and remarks

We can replace the integral condition for the existence of blow-up solutions to Eq. (1) with the function $g(t)$ given by (3) by the following series conditions:

Theorem 5.1 (*Series Sufficient Condition for the Existence of Blow-up*). *If the series*

$$\sum_{i=0}^{\infty} K^{-1}\left(t^{(1-q)(\frac{p}{q})^i}\right) \tag{11}$$

is convergent for some point $t \in (1, \infty)$, then a blow-up solution to (1) with function $g(t)$ given by (3) exists.

Theorem 5.2 (*Series Necessary Condition for the Existence of Blow-up*). *If Eq. (1) with function $g(t)$ given by (3) has a blow-up solution, then the series*

$$\sum_{i=0}^{\infty} K^{-1}\left(t^{(1-\varepsilon-p)(\frac{p}{r})^i}\right)$$

is convergent for any $t \in (1, \infty)$ and for any $\varepsilon \in (0, 1)$.

Proof. We can use for the sequence $t_0 = t$, $t_{n+1} = \Psi(t_n)$, where $\Psi(t) = t^{\frac{\alpha}{\beta}}$, $\alpha > \beta$ and $t > 1$, a technique from the proof of Theorem 2.1. We obtain then that for any $t > 1$

$$\ln\frac{\alpha}{\beta}\sum_{i=0}^{\infty} K^{-1}\left(t^{(1-\gamma)(\frac{\alpha}{\beta})^i}\right) \geq \int_t^{\infty} K^{-1}(s^{1-\gamma})\frac{\mathrm{d}s}{s(\ln s)} \geq \ln\frac{\alpha}{\beta}\sum_{i=1}^{\infty} K^{-1}\left(t^{(1-\gamma)(\frac{\alpha}{\beta})^i}\right), \tag{12}$$

where $\gamma > 1$. Because

$$\int_t^{\infty} K^{-1}(s^{1-\gamma})\frac{\mathrm{d}s}{s(\ln s)} = \int_0^{t^{1-\gamma}} K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s)}$$

for $t \in (1, \infty)$, hence the convergence of the integral $\int_t^{\infty} K^{-1}(s^{1-\gamma})\frac{\mathrm{d}s}{s(\ln s)}$, where $t \in (1, \infty)$, is equivalent to the convergence of the integral $\int_0^t K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s)}$, where $t \in (0, 1)$. From that fact, (12) and Theorem 2.1, we get immediately Theorems 5.1 and 5.2. ∎

Remark 5.3. Let us have in (1) the function $g(t) \sim t^p$ as $t \to \infty$, where $p > 1$. This means that there exist positive constants c, c_1, c_2 such that for any $t \in (c, \infty)$ we obtain $c_1 t^p \leq g(t) \leq c_2 t^p$. Suppose that we use the following fact:

Fact 1. *Let the functions $g_1(t)$, $g_2(t)$ satisfy conditions (2) and let $g_1(t) \geq g_2(t)$ for $t \in [0, \infty)$. If Eq. (1) with function $g(t) = g_2(t)$ has a blow-up solution that blows up in time T, then Eq. (1) with function $g(t) = g_1(t)$ also has a blow-up solution that blows up in at most time T.*

On the basis of Theorem 2.1 we can formulate the following theorem.

Theorem 5.4. *Let the function $g(t)$ satisfy conditions (2) and let $g(t) \sim t^p$, where $p > 1$. If the integral*

$$\int_0^t K^{-1}(s)\frac{\mathrm{d}s}{s(-\ln s)} \tag{13}$$

is convergent for some $t \in (0, 1)$, then a blow-up solution to (1) with function $g(t) \sim t^p$, where $p > 1$, exists.

6. Some examples

Example 6.1. Let in Eq. (1) the function $g(t)$ be given by (3) and the kernel $k(t)$ have the form $k(t) = \alpha t^{-\alpha-1}\exp(t^{-\alpha} - \exp(t^{-\alpha}))$, where $\alpha > 0$. We have then $K(t) = \exp(-\exp(t^{-\alpha}))$ and $K^{-1}(t) = 1/(\ln\ln(1/t))^{1/\alpha}$. Hence the integral (4) has the following form:

$$\int_0^t \frac{\mathrm{d}s}{s(-\ln s)(\ln\ln(1/s))^{1/\alpha}}. \tag{14}$$

When $\alpha \geq 1$, then the integral (14) is divergent for any $t \in (0, 1)$. This means on the basis of Theorem 2.1 that in Eq. (1) blow-up does not occur. On the other hand, when $0 < \alpha < 1$, then the integral (14) is convergent for any $t \in (0, 1)$ and from Theorem 2.1 we obtain that a blow-up solution to (1) exists.

Example 6.2. Now we consider Eq. (1) with the function $g(t)$ given by (3) and with the kernel $k(t) = t^{\alpha-1}$, where $\alpha > 1$. Then $K(t) = \frac{1}{\alpha}t^{\alpha}$, $K^{-1}(t) = (\alpha t)^{\frac{1}{\alpha}}$ and this implies that the integral (4) has the following form:

$$-\alpha^{\frac{1}{\alpha}}\int_{-\infty}^{\ln t}\frac{\mathrm{e}^{\frac{s}{\alpha}}}{s}\mathrm{d}s. \tag{15}$$

For $\alpha > 1$ the integral (15) is convergent for any $t \in (0, 1)$; hence on the basis of Theorem 2.1 this means that the Eq. (1) has a blow-up solution.

7. Final remark

In [3] it was shown that some nonlinear parabolic problems related to explosion can be converted to the integral Eq. (1) with power nonlinearity $k(x) = x^{-\frac{1}{2}}$. In this case $K(x) = 2x^{\frac{1}{2}}$ and $K^{-1}(x) = \frac{x^2}{4}$. It is easy to see that the integral (4) is equivalent to

$$\frac{1}{4}\int_0^t \frac{s}{-\ln s}\mathrm{d}s. \tag{16}$$

But the integral (16) is convergent for any $t \in (0, 1)$. This implies that there is a blow-up. This covers results presented earlier in [3].

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